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Symmetry relations in exactly soluble models

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Abstract. Symmetry relations such as the star-triangle or the inverse relation are very useful in determining the partition function of two-dimensional exactly soluble models. A common construction of the three-dimensional equivalents of these symmetry relations is presented here. They are used to derive, in a geometric way applying simultaneously to different kinds of spin models, the consequent global properties, i.e. the commutativity of the transfer matrices and the inverse functional equations on the transfer matrix and the partition function. The usefulness of the inverse relation is illustrated by an application to the three-dimensional Ising model.

1. Introduction

Symmetry relations such as the star-triangle relation and the inverse relation have been successfully exploited in different models of two-dimensional statistical physics (Ising models (Onsager 1971, Baxter and Enting 1978), vertex models (Baxter 1972, 1973, 1978, 1980b, Stroganov 1979), the critical Potts model (Baxter *et al* 1978), the hard-hexagon model (Baxter 1980a)). Schematically, these are relations which one imposes on the local statistical weights of the model, thus implying global properties like the commutativity of transfer matrices (Onsager 1971, Baxter 1972, 1973, 1978, 1980b, Baxter *et al* 1978) or functional equations on the partition function (Stroganov 1979, Baxter 1980a, b) or on some correlation functions (Baxter and Enting 1978), so that one can derive analytical expressions for these functions. These relations have also been used in two-dimensional particle physics to determine the S-matrix of particular models (Zamolodchikov 1979, Zamolodchikov and Zamolodchikov 1979) or the energy spectrum of some field theories (Faddeev *et al* 1979, Kulish and Sklyanin 1981).

It is an attractive challenge to try to extend to higher dimensions the class of the so-called exactly soluble models and their corresponding techniques. A generalisation of the triangle relation to a tetrahedron relation has been studied by Zamolodchikov (1981) in a three-dimensional model of factorised S-matrices and has also been used by Bazhanov and Stroganov (1981) to show the commutativity of transfer matrices in d dimensions. Unfortunately, the rank and the number of variables of the set of equations expressing the d-simplex relation increase very rapidly with the dimension, so that the explicit resolution rapidly becomes problematic. In this context, one has either to develop methods to uniformise the symmetry relations or to look for relations which

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are more easily exploitable and which still allow the deduction of global properties or functional equations on the partition function or the correlation functions. In this latter case, it appears helpful to take advantage of the geometric properties of the symmetry relations.

This article will precisely deal with this last approach. In § 2 we recall the definitions of the concepts which we shall use, in particular the product of statistical weights and the transfer matrix. In § 3, we describe the successive steps which enter the construction of a symmetry relation. The tetrahedron relation first illustrates the construction, which we then enlarge to include the case of the inverse relation. We use them in § 4 to derive, in a geometric way applying simultaneously to different kinds of spin models, the global properties we are interested in, i.e. the commutativity of transfer matrices and the inverse functional equations. Finally, in order to show that one can effectively exploit three-dimensional relations, we very briefly give an application of the inverse relation to the perturbative determination of the partition function of the three-dimensional Ising model.

2. Definitions

Consider a three-dimensional cubic lattice of size $L \times M \times N$. Spins will be localised on points, links, faces and volumes indifferently and will take as many values as there are different states for the corresponding elementary part of the system. As we simultaneously envisage the different kinds of spin models in which the elementary interactions occur between all the spins surrounding a point or included in a volume (figure 1), we shall denote by interaction star the set of all the points, links, faces, volumes

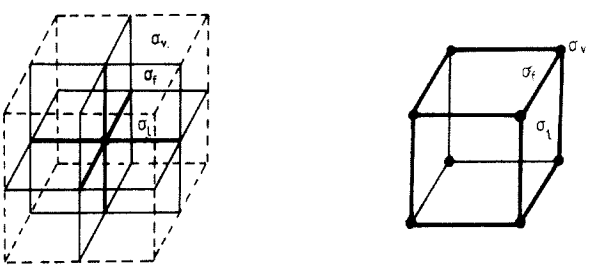

Figure 1. Interaction star: each figure represents all the points, links, faces, volumes and the corresponding spins which enter an elementary interaction; l, f, v respectively index the six links (or faces), the 12 faces (or links) and the eight volumes (or points); $w([\sigma_l]; [\sigma_f]; [\sigma_v])$ is the corresponding local statistical weight.

where the spins entering an elementary interaction are localised. Accordingly, to each configuration of the spins localised in an interaction star i, i.e. $[\sigma]_i$, there will correspond an interaction energy $E([\sigma]_i)$ and in consequence a local statistical weight $w([\sigma]_i) = \exp(-\beta E([\sigma]_i))$ (where β is the inverse of the absolute temperature) (figure 1). This is the set of parameters that will characterise the model. The partition function of the model is then given by

$$Z = \sum_{\{\sigma\}} \exp(-\beta E(\{\sigma\})) \tag{1}$$

where the summation is over all configurations of all spins and the corresponding energy is given by

$$E(\{\sigma\}) = \sum_{i \in I} E([\sigma]_i) \tag{2}$$

where the summation is over all interaction stars. The partition function can also be written as

$$Z = \sum_{\{\sigma\}} \prod_{i \in I} w([\sigma]_i). \tag{3}$$

This expression can be considered as the product of all the local statistical weights, provided one defines an elementary product in the following way (figure 2). Consider the two local weights w and w' associated with two neighbouring interaction stars i and i':

(1) glue the two stars identifying the spins on the common points, links, faces, volumes;

(2) to any configuration of the spins located in this block of stars, associate the product of the corresponding values of the local weights;

(3) take the sum of all the results given by all the different configurations of the spins which are located strictly inside the block (those which do not interact with the exterior any more). The final result is the statistical weight of the spins located on the boundary of the block. This is the desired product ww'.

An essential concept which is introduced in soluble models as a simplifying intermediate step to compute the partition function is that of transfer matrix. For that purpose, one cuts the lattice into similar $(d-1)$-dimensional slices (figure 3). One then takes the product, in the sense previously defined, of the local weights corresponding to all the interaction stars located inside a slice. The result is the statistical weight of the

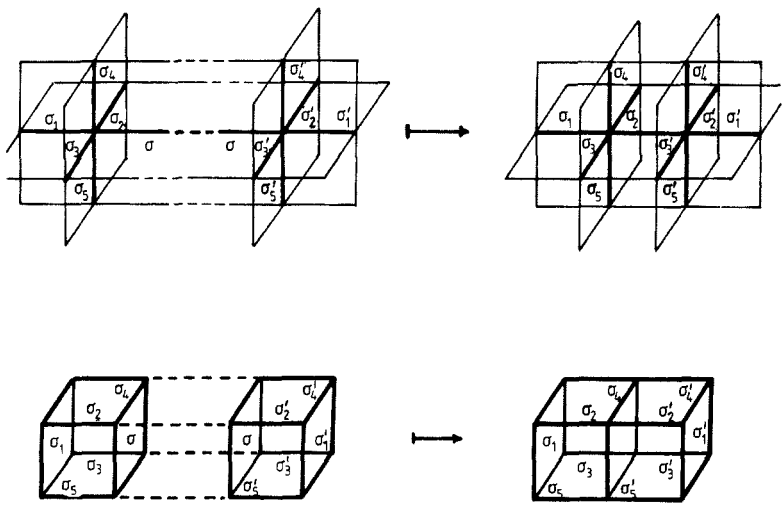

Figure 2. Product of two local weights:

$$\sum_{\sigma} w(\sigma_1, \sigma, \sigma_2, \sigma_3, \sigma_4, \sigma_5) w'(\sigma, \sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4, \sigma'_5)$$

$$= (ww')(\sigma_1, \sigma'_1, \sigma_2, \sigma'_2, \sigma_3, \sigma'_3, \sigma_4, \sigma'_4, \sigma_5, \sigma'_5).$$

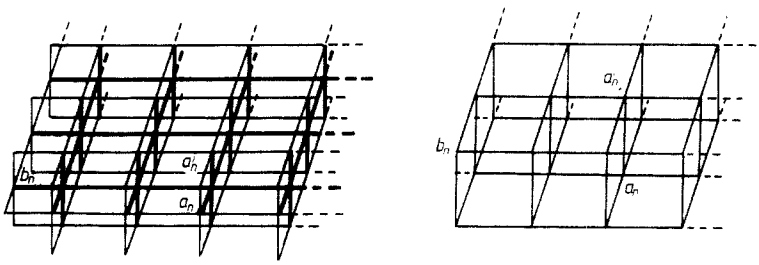

Figure 3. Transfer matrix: $T_b(a, a')$ with $a = (\ldots, a_i, \ldots)$, $a' = (\ldots, a'_i, \ldots)$, $b = (\ldots, b_j, \ldots)$.

slice, which is a function which associates a number to each configuration of the spins located on the boundary of the slice. Distinguishing between the lower and upper sides of the slice and considering each configuration of the spins on one side (a or a') as a basis vector for the vector space of states on that side, one can consider the statistical weight of the slice as a matrix which sends the vector space of states of the lower side into the vector space of states of the upper side (figure 3):

$$T_b(a, a') \qquad \text{with} \quad a = (\ldots, a_i, \ldots) \qquad i \text{ taking } L \times M \text{ values}$$
$$a' = (\ldots, a'_i, \ldots)$$
$$\text{and} \quad b = (\ldots, b_j, \ldots) \qquad j \text{ taking} \sim (L + M) \text{ values}$$

(the spins on the perimeter can be considered as additional parameters). This matrix is called the transfer matrix. The partition function is then given by the product of N similar transfer matrices (figure 4). With the preceding matrix interpretation, this corresponds to the operator product, and, with the usual periodic boundary conditions, to the trace operation of ordinary matrices. We shall prefer to use the partition function per site, i.e. $Z^{1/LMN}$, which we shall denote for simplicity by Z in the following. Hence

$$Z = [\mathrm{Tr}(T^N)]^{1/LMN}, \tag{4}$$

leading in the thermodynamic limit to

$$Z \simeq \lambda^{1/LM} \tag{5}$$

where λ is the largest eigenvalue of the transfer matrix T.

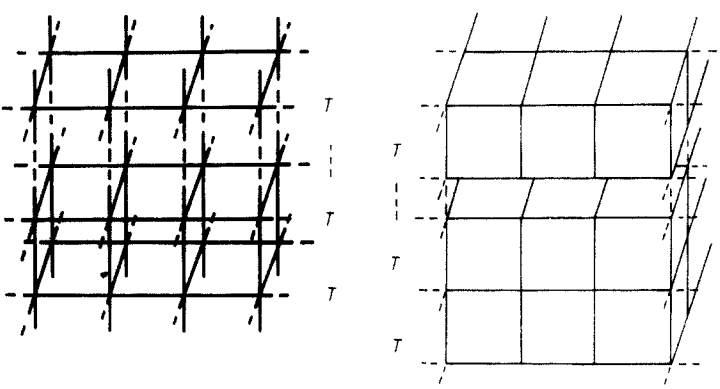

Figure 4. Partition function.

block: (1) separate the interaction stars, (2) rearrange them, (3) glue them back together in this new arrangement. This new arrangement must be chosen in such a way as to preserve the boundary of the block (the block boundary is a set of points, links, faces and volumes with definite neighbourhood relations and all this must remain unchanged). Let us illustrate this by the tetrahedron transformation and its dual image (in the geometric sense (Wegner 1971)): four interaction stars (four vertices or four parallelepipeds, respectively) are glued together to build a tetrahedron with its neighbourhood or a rhombic dodecahedron, respectively (figure 5). One obtains the second arrangement either by moving any of the four planes of the tetrahedron through the

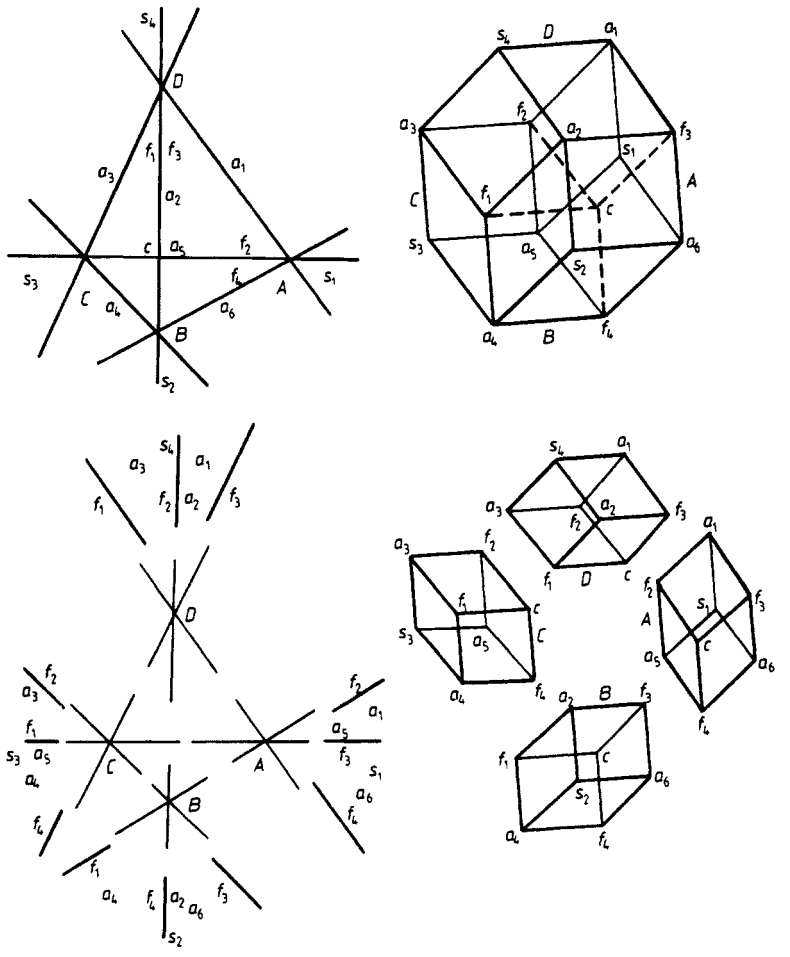

Figure 5. Decomposition of a tetrahedron (of a rhombic dodecahedron).

opposite vertex or by taking the symmetric image of the partitioned rhombic dodecahedron relatively to its centre (figure 6). These two arrangements build two blocks which both correspond to a tetrahedron (either upside up or upside down) with its neighbourhood or respectively to a rhombic dodecahedron with an inner star (which is either upside up or upside down). From one arrangement to the other, the boundary of the block does not change: this can easily be checked for the neighbourhood relations of the 12 links, 24 faces and 14 volumes surrounding the tetrahedron as well as for the 14 points, 24 links and 12 faces of the rhombic dodecahedron. The only change is in the inside, the tetrahedron itself or the inner star. Each of the two arrangements represents the gluing of the same four different interaction stars: to recognise the same star from one arrangement to the other, one can either follow the intersections of the planes as they move or take the symmetric image relatively to the centre of the rhombic dodecahedron. Let us now describe the symmetry relation which is involved in the preceding transformation. The two arrangements give two ways of gluing the interaction stars, hence two different blocks with the same boundary. As the two blocks have

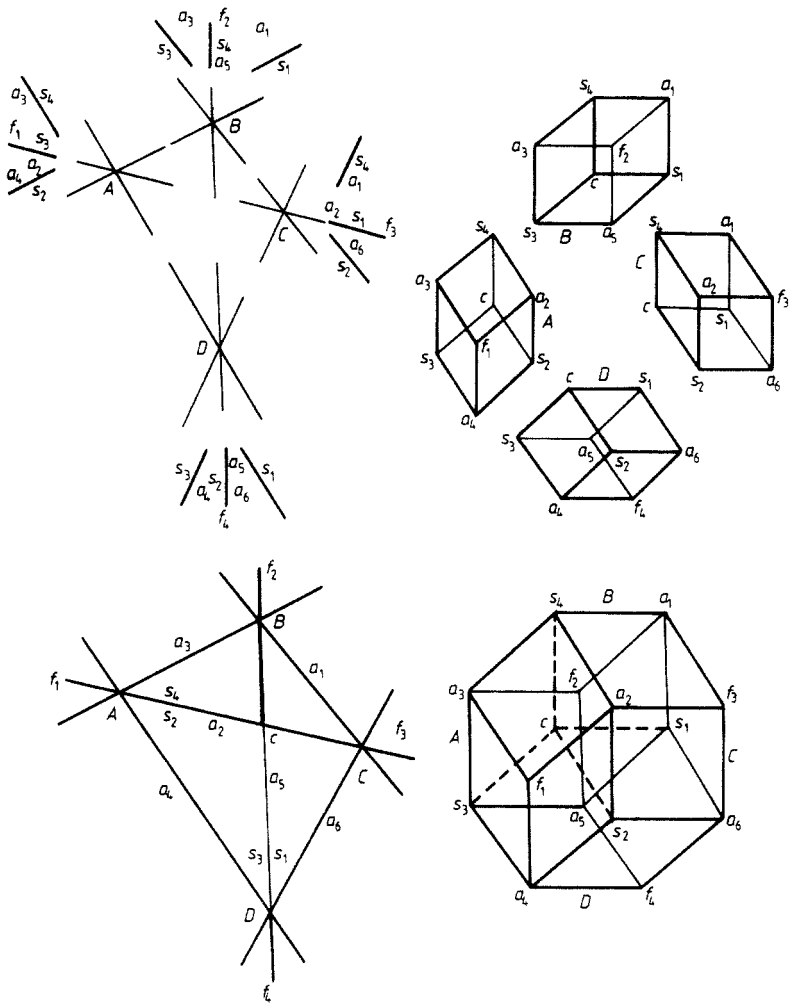

Figure 6. Rearrangement of the tetrahedron (of the rhombic dodecahedron).

the same boundary, their statistical weights will be functions defined on the same set and thus will be of the same form. But, as the local weights of the stars are different, the statistical weights of the blocks will *a priori* be different, i.e. there will exist configurations of the spins for which they are *a priori* different. Now, we shall impose that the statistical weights of the two blocks be indeed equal, for any configuration of the boundary spins, and this will be the symmetry relation. Let us make explicit this relation in detail, considering, for simplicity, the case of a model with volume spins only (the general case is derived in the same way). Associate a spin with each volume in or around the tetrahedron, figures 5 and 6, summing up the conventions: c the central spin, s_i $(i = 1, \ldots, 4)$ the four spins behind a vertex, a_i $(i = 1, \ldots, 6)$ the six spins behind a link and f_i $(i = 1, \ldots, 4)$ the four spins behind a face. The spin dependence of the different interaction stars and of their corresponding local weights (A, B, C, D) is indicated by figures 5, 6 and 7 for each of the two arrangements. Equating the two respective products gives the equation expressed by figure 7, that is the tetrahedron relation

$$\begin{aligned}\sum_c A(c, f_2, f_4, f_3, a_5, a_1, a_6, s_1)B(c, f_3, f_4, f_1, a_6, a_2, a_4, s_2)\\ \times C(c, f_4, f_2, f_1, a_5, a_4, a_3, s_3)D(c, f_1, f_2, f_3, a_3, a_2, a_1, s_4)\\ = \sum_c A(c, s_2, s_3, s_4, a_4, a_2, a_3, f_1)B(c, s_3, s_1, s_4, a_5, a_3, a_1, f_2)\\ \times C(c, s_4, s_1, s_2, a_1, a_2, a_6, f_3)D(c, s_1, s_3, s_2, a_5, a_6, a_4, f_4).\end{aligned} \tag{6}$$

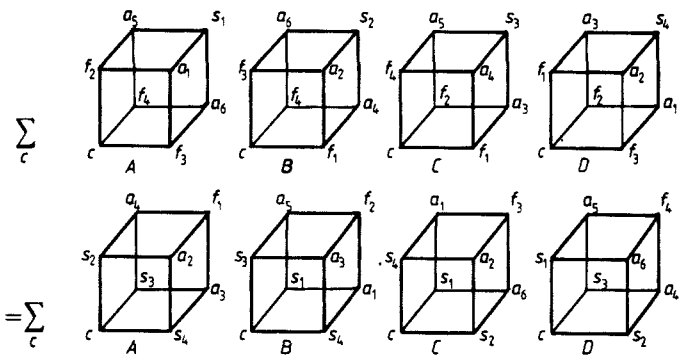

Figure 7. Tetrahedron relation.

One can make the following observations on this set of equations. The variables are the values of the local weights for each configuration of the spins $(\exp(-\beta E([\sigma]_i)))$ and constitute four groups $(A([\sigma]), B([\sigma]), C([\sigma]), D([\sigma]))$. As each interaction star depends on eight spins, each spin taking n different values, each group contains n^8 variables. Since the equality holds for any value of the 14 boundary spins, the tetrahedron relation is expressed by a set of n^{14} homogeneous algebraic equations in the $4n^8$ variables, linear in each variable separately, but of overall degree four. Such a system is overdetermined and constraints must be imposed on the variables of each local weight (A, B, C, D) to allow for a solution (these constraints come from the elimination of the other groups of variables and lead to algebraic subvarieties). Clearly, the large number of variables and equations makes the explicit resolution of the

tetrahedron relation and the corresponding parametrisation of the local weights difficult. Let us remark that the symmetry relations corresponding to the boundary preserving transformation of any block of interaction stars are obtained in the same way (for instance, one can envisage any kind of zonohedron (see Coxeter 1973) with its central symmetry).

Consider, next, a more general type of local transformation where the number of interaction stars is not conserved. Instead of rearranging a block of stars, we shall try to insert such a block into a lattice. Consider for instance the block of two interaction stars which one obtains either by joining two vertices at three lines or by joining two parallelepipeds along three faces (figure 8). This block can also be seen as the result

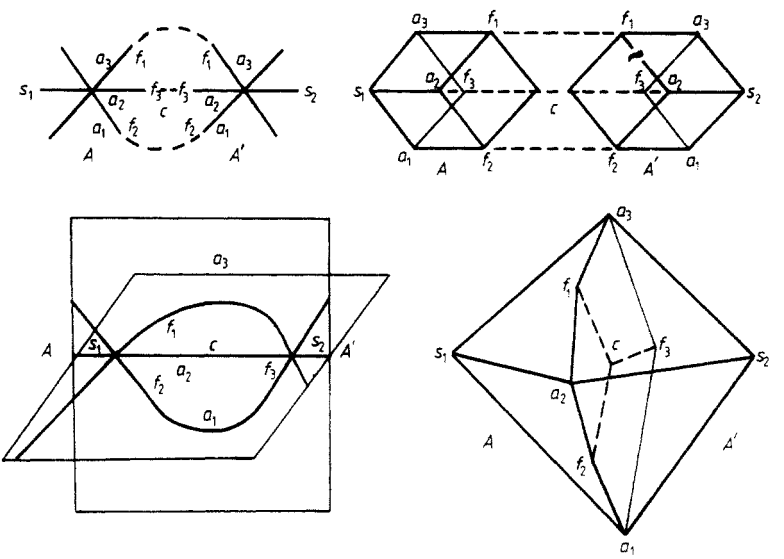

Figure 8. Inversion block.

of the double intersection of a surface with two planes, or as its dual image. In order to insert such a block into the lattice, one first needs to make a hole, that is, to duplicate some points, links, faces, volumes (and of course the corresponding spins) and to pull these elements apart, so that one can introduce the block into the available space (figure 9). This hole will have no effect on the lattice, if it is associated with a statistical weight which is equal to zero when any two duplicates of a spin are different and equal to a constant when all duplicates are equal (this is a kind of identity operator). The insertion of the block is the transformation which replaces the hole by the block. This transformation will be a symmetry if one imposes the equality of the two statistical weights, that is, if the statistical weight of the block (the product of the local weights) is either zero or a constant, according to whether any two spins which otherwise would identify are different, or whether all such pairs of spins are effectively equal. Thus, the inserted interaction stars with their local weights can be considered as the inverse of each other and the symmetry relation which is involved by the previous transformation is called the inverse relation. Let us now give the inverse relation in detail, reconsidering for simplicity a model with volume spins only. The spin conventions are summed up by figure 8: c the central spin, s_i $(i = 1, 2)$ the spins behind a vertex, a_i $(i = 1, 2, 3)$ behind a

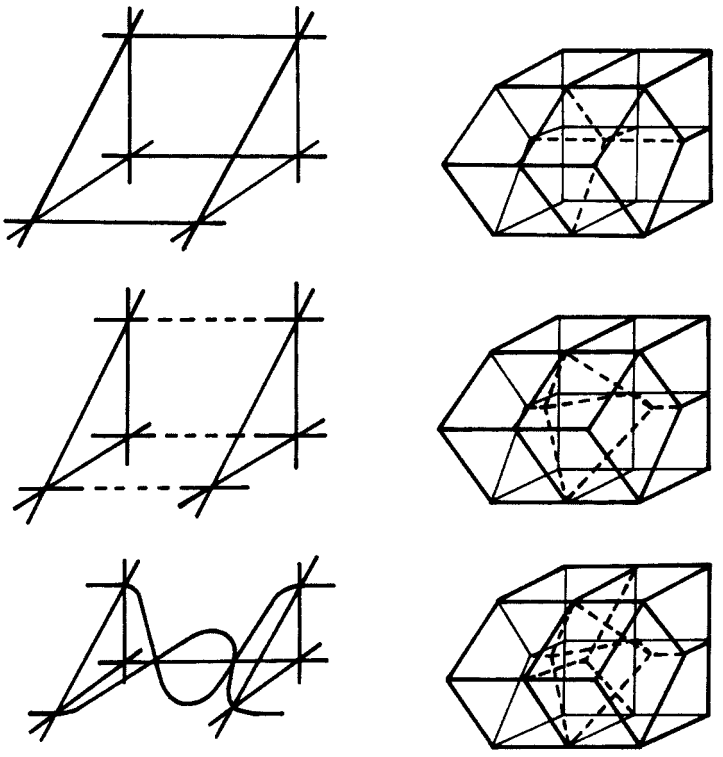

Figure 9. Insertion of the inversion block.

link, f_i ($i = 1, 2, 3$) behind a face. The spin dependence of the local weights is given by figure 10 which also expresses the inverse relation:

$$\sum_c A(s_1, a_3, a_1, a_2, f_1, f_2, f_3, c)A'(c, f_1, f_3, f_2, a_3, a_2, a_1, s_2) = \Lambda\delta_{s_1 s_2}. \quad (7)$$

Let us examine this set of equations in more detail. The variables are again the different values of the local weights and constitute two groups (A, A') of n^8 variables, with an additional variable Λ. The equality must hold for any values of the spins on the boundary and thus provides n^8 equations. For a given weight A, it allows us to determine, through the resolution of a linear system, another local weight A', which is its inverse. In practical cases, *a priori* constraints are imposed on the local weights (symmetries, normalisation, exclusions, etc) and the counting is different, but the resolution of the inverse relation remains simpler than that of the tetrahedron relation. The inverse relation only imposes that the *a priori* constraints be compatible with the

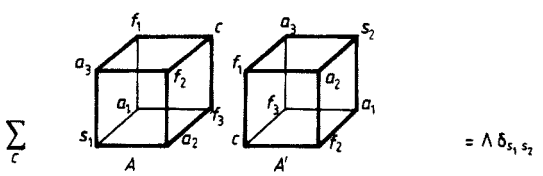

Figure 10. Inverse relation.

existence of an inverse satisfying the same constraints (the normalisation factor Λ may thus be determined by the local weight A). Furthermore, one can check the existence of the inverse relation on two-dimensional Ising models (non-planar or with field), on d-dimensional Potts models, percolation models, etc.

4. Commuting transfer matrices—inverse functional equations

After constructing the local symmetry relations, we shall apply them to derive global properties for the model. Let us first recall how to visualise both of these transformations: for the tetrahedron relation, four planes intersect three at a time and then one of them moves through the intersection of the other three (figures 5 and 6); for the inverse relation, one surface is locally parallel to the intersection of two planes and is then pushed through this intersection, cutting it twice (figures 8 and 9). Let us make the following remark: if one considers, instead of a regular cubic lattice, a non-regular lattice made of arbitrary surfaces, restricted to behave like planes at large distances, then the previous transformations generate the deformations of any surface, while keeping all the relative asymptotic positions fixed. Then, if one can associate local statistical weights at all the intersections, in such a way that the tetrahedron and inverse relations are satisfied everywhere on the lattice, the previous transformations will leave the partition function unchanged. This will thus constitute a generalisation of the Z invariance (Baxter 1972, 1973, 1978, Zamolodchikov 1981), where the planes are allowed not only to move but also to deform themselves.

Coming back to a regular lattice, we shall now deduce the commutativity of transfer matrices from the tetrahedron relation. Take the product of two transfer matrices, one with all its local weights equal to a given weight A, and the other with all its weights equal to another weight B (figure 11). First modify this product in the following way: make the two slices which build the two transfer matrices intersect along one half of their perimeter, or more precisely multiply the product of the two transfer matrices by a set of identical weights C all along one side of the perimeter and by a set of other identical weights D along an adjacent side (figure 12). Suppose now that the four different local weights A, B, C, D introduced so far (one for each transfer matrix, one for each side) satisfy the tetrahedron relation. One can begin to make a tetrahedron transformation at the corner where these four weights touch one another. The result is two-fold (figure 13(a)): on the one hand, the two transfer matrices have been exchanged at this corner; on the other hand, two other blocks of four weights satisfying the tetrahedron relation have been created. One can then repeat the transformation and have it propagate along the sides of the perimeter (figure 13(b, c, d)). At the end, one remarks that, while interchanging the two transfer matrices along two lines on the edge, one has also recovered the initial situation, with a corner of four weights satisfying

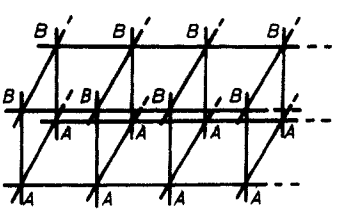

Figure 11. Product of two transfer matrices.

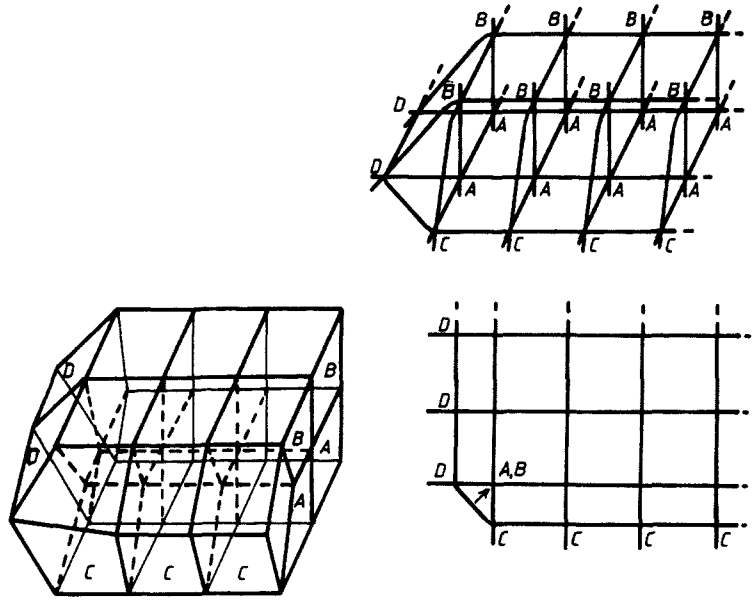

Figure 12. Crossed product of two transfer matrices: direct, dual and projected images.

the tetrahedron relation (figure 13(*d*)). The previous transformations can be repeated, line after line, until one finally has the following situation: the slices of the two transfer matrices are completely interchanged and the modification on the perimeter has moved, unchanged, to the opposite side (figure 13(*e*, *f*)). Recalling the matrix interpretation, one sees that the product of two transfer matrices taken in a given order has been transformed into the product of the same matrices, but taken in the opposite order, and simultaneously modified by a matrix product on the perimeter spins corresponding to a conjugation:

$$\sum_{c,c'} S^{C,D}(b, b'; c, c') T^{A}_{c,b''} T^{B}_{c',b'''} = \sum_{c'',c'''} T^{B}_{b',c'''} T^{A}_{b,c''} S^{C,D}(c'', c'''; b'', b''') \tag{8}$$

or

$$\sum_{\substack{c,c' \\ c'',c'''}} S^{C,D}(b, b'; c, c') T^{A}_{c,c''} T^{B}_{c',c'''} [S^{C,D}(c'', c'''; b'', b''')]^{-1} = T^{B}_{b',b'''} T^{A}_{b,b''}. \tag{9}$$

Introduce the periodic transfer matrix, that is the transfer matrix one obtains by imposing periodic boundary conditions:

$$\tau = \mathrm{diag}_{b_e} \mathrm{Tr}_{b_i} T_{b_e b_i} \tag{10}$$

where $\mathrm{diag}_{b_e} \mathrm{Tr}_{b_i}$ means that opposite spins (both external b_e and internal b_i) have been identified, and that the sum over all configurations of internal spins b_i has been taken. Identifying opposite spins in equalities (9) and summing over all configurations of the internal spins b_i and the external spins b_e common to both transfer matrices, one obtains the commutativity of the periodic transfer matrices:

$$\tau^A \tau^B = \tau^B \tau^A. \tag{11}$$

(Let us note that a parametrisation of the symmetry relation will provide families of

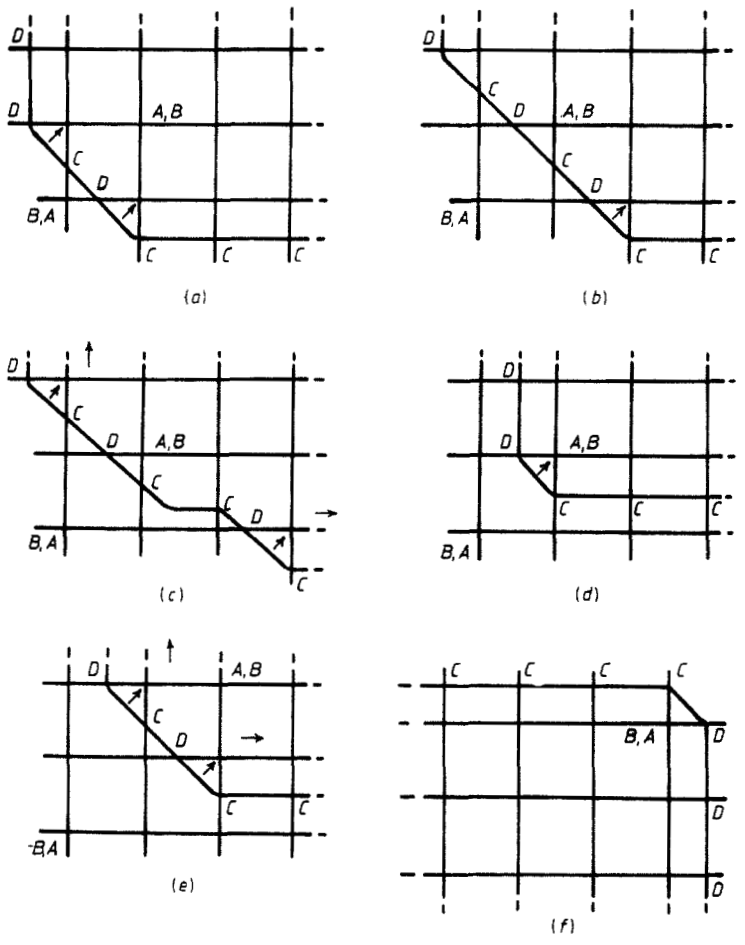

Figure 13. Sequence of tetrahedron transformations leading to the commuted product.

models satisfying the symmetry relation and thus having commuting periodic transfer matrices; that is, if u is a parameter varying inside the family

$$[\tau(u), \tau(v)] = 0 \qquad \text{for any } u, v. \tag{12}$$

Developing around a point ($u = 0$ for instance),

$$\tau(u) = \sum_j \tau_j u^j \tag{13}$$

will give an infinite set of operators in involution:

$$[\tau_i, \tau_j] = 0 \qquad \text{for any } i, j.) \tag{14}$$

Let us now deduce the similar global properties for the inverse relation. For that purpose, we shall follow quite faithfully the preceding sequence of transformations. Consider the product of two transfer matrices, each one built out of the same local weights, either A or A', these two weights satisfying the inverse relation. Contract all pairs of lines or, in the dual image, all pairs of faces along two sides of the perimeter, identifying and summing over the corresponding spins (figure 14). One can make the

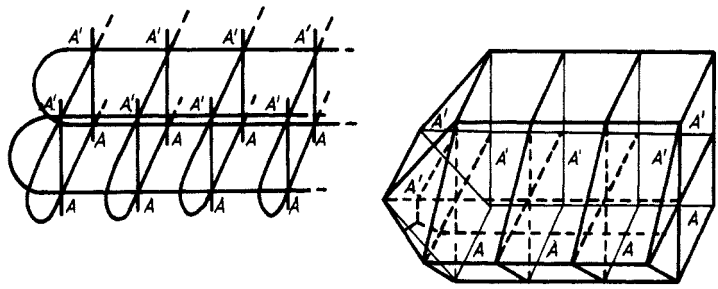

Figure 14. Pinched product of two transfer matrices: direct and dual images.

inverse transformation at the corresponding corner (figure 15(a)). The result is again two-fold: an identity operator at the corner and two contractions on the sides (figure 15(b)). The inverse transformation can then be applied along the two sides (figure 15(c, d, e)) and then again at the corner (figure 15(f)) and so on (figure 15(g)). At the end, this gives identity operators all over the slices of the transfer matrices (figure 15(h)). Considering the initial contraction as an additional product on the perimeter spins, one obtains the inverse functional equation for transfer matrices:

$$T^{A}T^{A'} = \Lambda^{LM}(A)\mathbb{1} \tag{15}$$

or else

$$\sum_{b''} T^{A}_{b'',b}T^{A'}_{b'',b'} = \Lambda^{LM}(A)\mathbb{1}\delta_{bb'}. \tag{16}$$

To illustrate the type of constraints this relation imposes on the model, let us consider the case of a one-parameter family. After proper normalisations

$$T(u)T(-u) = \mathbb{1}. \tag{17}$$

Developing around a point ($u = 0$) gives an infinite set of operator relations which look very similar to those of a set of operators in involution ({ , } is the usual anticommutator):

$$T_0^2 = \mathbb{1}, \qquad [T_0, T_1] = 0, \qquad \{T_0, T_2\} = T_1^2, \qquad [T_0, T_3] = [T_1, T_2],$$
$$\{T_0, T_4\} + T_2^2 = \{T_1, T_3\}, \qquad [T_0, T_5] = [T_1, T_4] + [T_3, T_2], \tag{18}$$

. . . .

Recalling that, in the thermodynamic limit, the partition function per site is given by the largest eigenvalue of the periodic transfer matrix, the inverse relation on the transfer matrix will heuristically provide the similar functional equation on the partition function per site:

$$Z(A)Z[I(A)] = \Lambda(A) \tag{19}$$

(where A is the set of parameters representing a model, $I(A)$ the set representing the inverse model and $\Lambda(A)$ the normalising factor). In fact, the derivation of this functional equation is not straightforward. (For a precise discussion of this derivation see Jaekel and Maillard (1981).) First the limit of an infinite lattice must allow us to neglect the effects on the perimeter spins induced by the contractions and to replace the

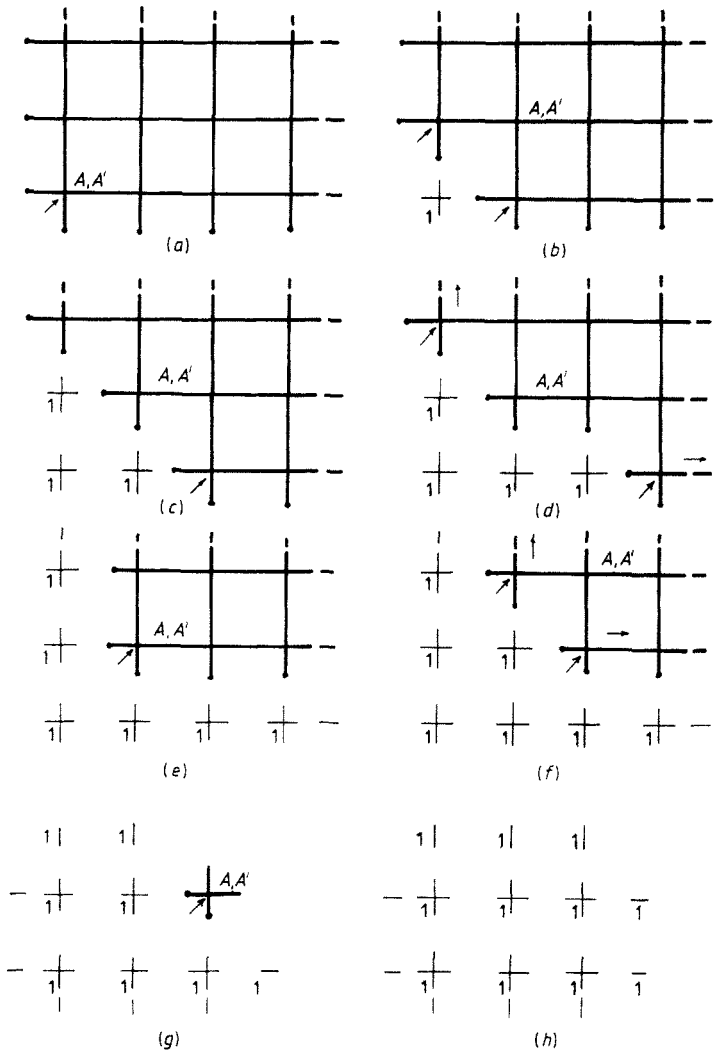

Figure 15. Sequence of inverse transformations leading to the identity operator.

transfer matrix T in equation (15) by the periodic transfer matrix τ, giving

$$\tau^{A}\tau^{I(A)} = \Lambda^{LM}(A)\mathbb{1}. \tag{20}$$

Then this equality holds between the largest eigenvalue of τ^{A} and the smallest eigenvalue of $\tau^{I(A)}$, and therefore will not relate the partition function calculated at A to the partition function calculated at $I(A)$. It seems natural to consider that equality (19) holds between the partition function at A and an analytical continuation at $I(A)$. This has been checked to be true in one or two dimensions and constitutes a conjecture in d dimensions. Our discussion below also provides a first step towards its verification for the three-dimensional Ising model.

To conclude this section, we give a brief account of an application of the inverse relation to the three-dimensional Ising model. Let $A = (K_1, K_2, K_3)$ be the standard

interaction constants in the three different directions of the cubic lattice. The inverse relation then gives

$$I(K_1, K_2, K_3) = (K_1 + \tfrac{1}{2}\mathrm{i}\pi, -K_2, -K_3), \qquad \Lambda = 2\mathrm{i}\sinh 2K_1 \tag{21}$$

and for the inverse functional equation on the partition function per site

$$Z(K_1, K_2, K_3)Z(K_1 + \tfrac{1}{2}\mathrm{i}\pi, -K_2, -K_3) = 2\mathrm{i}\sinh 2K_1. \tag{22}$$

Introducing the more convenient parameters

$$u = \tanh K_1, \qquad v = \tanh K_2, \qquad w = \tanh K_3, \tag{23}$$

the inverse relation becomes

$$Z(u, v, w)Z(1/u, -v, -w) = 4\mathrm{i}u/(1-u^2). \tag{I}$$

One also has the obvious symmetry between the three directions

$$Z(u, v, w) = Z[\sigma(u, v, w)] \tag{S}$$

where σ is any permutation of the three variables, u, v, w. After extracting a normalising factor

$$Z(u, v, w) = 2[(1-u^2)(1-v^2)(1-w^2)]^{-1/2} z(u, v, w) \tag{24}$$

and taking the log, we shall use the partially resummed high-temperature expansion (Baxter 1980b, Jaekel and Maillard 1981)

$$\ln z(u, v, w) = (v^2 + w^2)f_1(u) + v^2w^2 f_2(u) + (v^4 + w^4)f_3(u) + \ldots \tag{25}$$

where f_1, f_2 and f_3 are rational functions of u, resulting from the partial resummation. One can see that f_1, f_3 are equal to the corresponding two-dimensional functions and that f_2 has only $(1-u^2)^3$-type singularities:

$$f_1(u) = \frac{u^2}{1-u^2}, \qquad f_2(u) = \frac{a + bu^2 + cu^4 + du^6}{(1-u^2)^3}, \qquad f_3(u) = \frac{u^2}{(1-u^2)^3}(1 - \tfrac{1}{2}u^2 + \tfrac{1}{2}u^4). \tag{26}$$

The inverse (I) and symmetry (S) relations on z are

$$\ln z(u, v, w) + \ln z(1/u, -v, -w) = \ln(1-v^2)(1-w^2), \tag{I}$$

$$\ln z(u, v, w) = \ln z[\sigma(u, v, w)]. \tag{S}$$

From these equations and from the knowledge of the coefficient of the $u^2v^2w^2$ term which is 16, one can immediately determine f_2:

$$f_2(u) = (1 + 13u^2 + 13u^4 + u^6)/(1-u^2)^3. \tag{27}$$

We have also obtained this result without using (I) and (S) by determining all the corresponding diagrams, which are shown in figure 16. Note that the standard high-temperature expansion can be recovered from the preceding result, up to the eighth order:

$$\begin{aligned}\ln z(u, v, w) = {} & u^2v^2 + u^2w^2 + v^2w^2 + 16u^2v^2w^2 + u^4v^2 + u^4w^2 + v^4u^2 + w^4u^2 \\ & + v^4w^2 + w^4v^2 + 58(u^4v^2w^2 + v^4u^2w^2 + w^4u^2v^2) \\ & + \tfrac{5}{2}(u^4v^4 + u^4w^4 + v^4w^4) + u^6v^2 + u^6w^2 \\ & + v^6u^2 + w^6u^2 + v^6w^2 + w^6v^2 + \ldots\end{aligned} \tag{28}$$

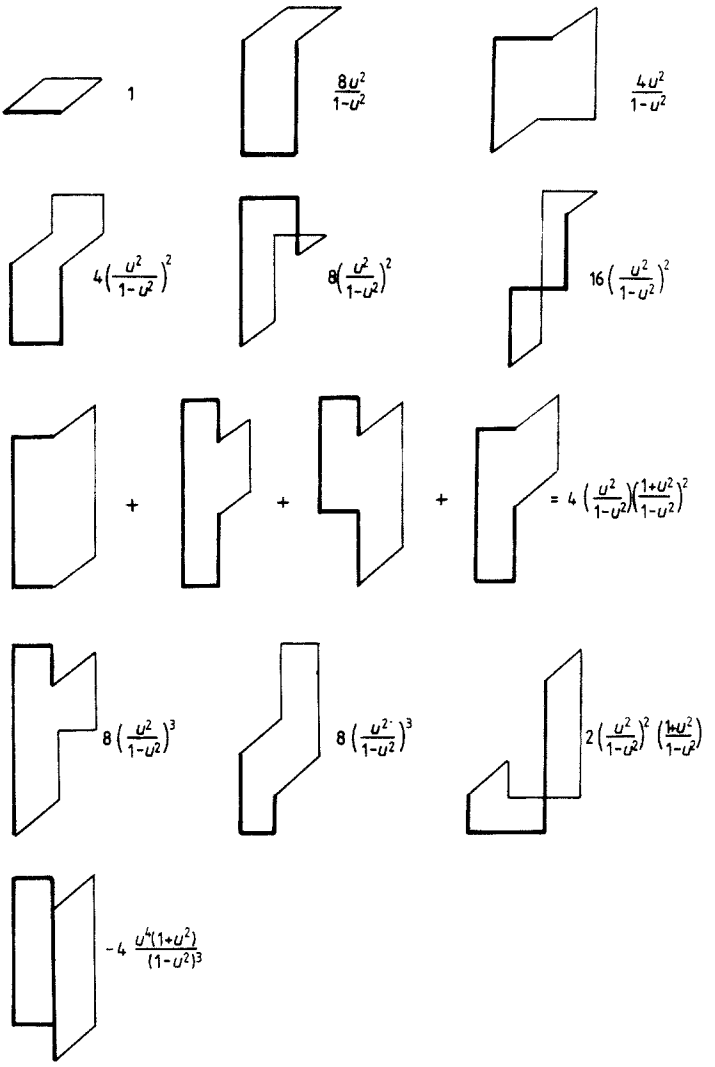

Figure 16. Diagrams of order v^2w^2 in the three-dimensional Ising model.

(in agreement with the isotropic case (Sykes *et al* 1972)). These procedures can be repeated to any order of the expansion and can then be interpreted in two ways: either as a verification of the inverse relation on the partition function, through the diagrammatic expansion of figure 16 (and thus the inverse relation can be used as a control test of the exactness of the result) or as a determination of the partition function through the relations (I) and (S) and a little more information on a particular type of diagram. A more extensive discussion of this problem is now under study.

5. Conclusion

In this article, we have tried to give a common construction of symmetry relations and also a common derivation of global properties for the transfer matrix or the partition

function. In particular, just as for the two-dimensional hard-hexagon model (Baxter 1980a), these properties can be derived for genuine spin models, without using a particular matrix formalism. Note that the inverse relation is satisfied by the whole transfer matrix, and thus provides the inverse functional equation not only for the largest eigenvalue but also for any eigenvalue. One can thus look for inverse functional equations not only on the partition function but also on the interface energy (as can be verified for the two-dimensional Ising model) or on some correlation lengths (Watson 1977). However, in this case, one loses the additional spatial symmetries which were satisfied by the partition function.

We have also indicated in the three-dimensional Ising case, how one can exploit the constraints which the inverse relation imposes on the partition function. In this approach, no connection was made with any star-triangle or tetrahedron relation, but instead, the inverse relation was combined with a particular diagrammatic expansion of the partition function. Such a procedure also applies to other models like d-dimensional Potts models (Jaekel and Maillard 1981) and two-dimensional non-planar (or with field) Ising models.

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